Linear Local Vol Cheyette PDE.

Let the PDE i want to solve,

$$\frac{\partial V_t}{\partial t} + \left(y_t - \kappa x_t\right)\frac{\partial V_t}{\partial x_t} + \frac{1}{2}\sigma^2(t, x_t)\frac{\partial^2 V_t}{\partial x_t^2} + \left(\sigma^2(t, x_t) - 2\kappa y_t\right)\frac{\partial V_t}{\partial y_t} = r_t V_t$$
$$\text{s.t} \qquad V(T, x_T, y_T) = \phi(T, x_T, y_T) \tag{1}$$

where the local volatility is the one proposes by Piterbar & Andreasen's book [Pit&And]

$$\sigma(t, x_t) = a_t(b_t + c_t x_t)$$

and the short rate has been denoted by

$$r_t = x_t + f(0, t)$$

being $f(0, t)$ the instantaneous forward rate with maturity t.
We apply the change of variable as proposed in [Pit&And]

$$u_t = y_t - \bar{y}_t \qquad \text{where} \qquad \bar{y}_t = E^{\mathbb{Q}}\left[y_t|\mathcal{F}_0\right]$$

So the PDE we solve for is

$$\frac{\partial V_t}{\partial t} + \mathcal{L}_x V_t + \mathcal{L}_u V_t = 0$$
$$\text{s.t} \quad V(T, x_T, y_T) = \phi(T, x_T, y_T) \tag{2}$$

where

$$\mathcal{L}_x = \left(u_t + \bar{y}_t - \kappa x_t\right)\frac{\partial}{\partial x_t} + \frac{1}{2}\sigma^2(t, x_t)\frac{\partial^2}{\partial x_t^2} - r_t$$
$$\mathcal{L}_u = \underbrace{\left(\left(\sigma^2(t, x_t) - \sigma^2(t, x_0)\right) - 2\kappa u_t\right)}_{\mu_u(t, x_t, y_t)}\frac{\partial}{\partial u_t} \tag{3}$$

Operator Discretization and boundary conditions

We approximate the PDE in (4) by the following system of ODEs:

$$\begin{aligned}&\frac{\partial V_t}{\partial t} + A_x V_t + A_u V_t = 0\\ &\text{s.t} \quad V(T, x_T, y_T) = \phi(T, x_T, y_T)\end{aligned} \tag{4}$$

where A_x and A_u denotes the discrete differential operator for $\mathcal{L}_x$ and $\mathcal{L}_u$ respectively, for $\{x_j\}_{j=0,\ldots,N}$ and $\{u_j\}_{k=0,\ldots,M}$.

x-Discretization:

We discretize first and second derivatives in x direction with 3-point centered differences. We can use an upwind scheme for the first derivative (depending on the value of the diffusion term relative to the drift one, but this does not change results very much). We assume that the second derivative goes to zero at both ends.

u-Discretization:

Two different approachs:

Upwind Scheme:

We approximate within the grid:

$$\left.\frac{\partial V}{\partial u}\right|_{j,k} \approx \frac{V_{j,k+1} - V_{j,k}}{\Delta_u}\mathbf{1}_{\{\mu_u(x_j,u_k)>0\}} + \frac{V_{j,k} - V_{j,k-1}}{\Delta_u}\mathbf{1}_{\{\mu_u(x_j,u_k)<0\}} \qquad \forall\ k = 1, \ldots, M-1$$

At the boundaries:

$$\left.\frac{\partial V}{\partial u}\right|_{j,0} = \frac{V_{j,1} - V_{j,0}}{\Delta_u}\mathbf{1}_{\{\mu_u(x_j,u_0)>0\}} + \frac{V_{j,0} - V_{j,-1}}{\Delta_u}\mathbf{1}_{\{\mu_u(x_j,u_0)<0\}}$$

where $V_{j,-1}$ is a value associated to a point that lies out of the grid. In order to express it in terms of the values within the grid, we assume that the second derivative goes to zero at the boundary. In the case of an uniform discretization,

$$\frac{\partial^2 V}{\partial u^2} = 0 \ \Rightarrow V_{j,-1} = 2V_{j,0} - V_{j,1}$$

So at the boundary we eventually have,

$$\left.\frac{\partial V}{\partial u}\right|_{j,0} = \frac{V_{j,1} - V_{j,0}}{\Delta_u}$$

independent of the sign of the convection term.

Following the same steps for the other boundary,

$$\left.\frac{\partial V}{\partial u}\right|_{j,M} = \frac{V_{j,M} - V_{j,M-1}}{\Delta_u}$$

Five Point stencil:

We approximate within the grid:

$$\left.\frac{\partial V}{\partial u}\right|_{j,k} \approx \alpha_{k,1} V_{j,k-2} + \alpha_{k,2} V_{j,k-1} + \alpha_{k,3} V_{j,k} + \alpha_{k,4} V_{j,k+1} + \alpha_{k,5} V_{j,k+2}, \qquad \forall\ k = 2, \ldots, M-2$$

For uniform grids,

$$\alpha_{k,1} = \frac{1}{12}\frac{1}{\Delta_u},\ \alpha_{k,2} = -\frac{2}{3}\frac{1}{\Delta_u},\ \alpha_{k,3} = 0,\ \alpha_{k,4} = \frac{2}{3}\frac{1}{\Delta_u},\ \alpha_{k,5} = -\frac{1}{12}\frac{1}{\Delta_u}$$

At the lower boundary ($k = 1$),

$$\left.\frac{\partial V}{\partial u}\right|_{j,1} = \alpha_{1,1} V_{j,-1} + \alpha_{1,2} V_{j,0} + \alpha_{1,3} V_{j,1} + \alpha_{1,4} V_{j,2} + \alpha_{1,5} V_{j,3} \tag{5}$$

Again, to get rid of the ghost point, we assume linear function at the boundary (assume equidistant grid to ease notation),

$$\left.\frac{\partial^2 V}{\partial u^2}\right|_{j,0} = 0 \quad \Rightarrow \quad V_{j,-1} = 2V_{j,0} - V_{j,1} \tag{6}$$

So equation (5) becomes,

$$\left.\frac{\partial V}{\partial u}\right|_{j,1} = (2\alpha_{1,1} + \alpha_{1,2})\, V_{j,0} + (\alpha_{1,3} - \alpha_{1,1})\, V_{j,1} + \alpha_{1,4} V_{j,2} + \alpha_{1,5} V_{j,3} \tag{7}$$

At the lower boundary ($k = 0$),

$$\left.\frac{\partial V}{\partial u}\right|_{j,0} = \alpha_{0,1} V_{j,-2} + \alpha_{0,2} V_{j,-1} + \alpha_{0,3} V_{j,0} + \alpha_{0,4} V_{j,1} + \alpha_{0,5} V_{j,2} \tag{8}$$

Again, to eliminate the ghost points,

$$\left.\frac{\partial^2 V}{\partial u^2}\right|_{j,-1} = 0 \quad \Rightarrow \quad V_{j,-2} = 2V_{j,-1} - V_{j,0} \tag{9}$$

Substituting (9) and (6) into (8), we obtain

$$\left.\frac{\partial V}{\partial u}\right|_{j,0} = V_{j,0}\left(3\alpha_{0,1} + 2\alpha_{0,2} + \alpha_{0,3}\right) + V_{j,1}\left(\alpha_{0,4} - 2\alpha_{0,1} - \alpha_{0,2}\right) + V_{j,2}\,\alpha_{0,5} \tag{10}$$

Time evolution

To evolve the PDE in time we apply the Craig Sneyd ADI method:

$$\Big[1 - \theta \Delta_t A_x\Big] U(t) = \Big[1 + (1-\theta)\,\Delta_t A_x + \Delta_t A_u\Big] V(t+\Delta) \tag{11}$$

$$\Big[1 - \theta \Delta_t A_u\Big] V(t) = U(t) - \theta \Delta_t A_u V(t+\Delta) \tag{12}$$

where in the case of the five point stencil, system (12) is a penta diagonal one.